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CONSTRUCTIONS FOR PLANAR POINT PROCESSES USING CONCENTRIC CIRCLES

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Some point processes are obtained by generalising the well-known construction for a two-dimensional Poisson process which locates an event on each of a sequence of concentric circles in a particular way. The constructions considered here have, in general, a random number of events on each circle. Under certain sufficient conditions, the constructed processes are asymptotically Poisson, far from the origin. The obvious regularity in the structure of these processes can be removed, at least superficially, by displacing the events independently off the concentric circles.

Multidimensional point process, Poisson process construction, Asymptotic Poisson behaviour.

1. Introduction

A comprehensive survey of point process theory can be found in [2] and while for the most part the processes are assumed to be in $\mathbf{R}^1$, the theory concerned with counting properties of the processes usually holds unchanged in $\mathbf{R}^n$. The theory specifically for a multidimensional process is described by Fisher [3].

However in $\mathbf{R}^1$ most constructions of point processes are made via the interval sequence and have no natural analogue in $\mathbf{R}^n$ $(n > 1)$. Thus there is a shortage of models for multidimensional point processes, those considered most often tend either to be based on the Poisson process, e.g. doubly stochastic Poisson process or Poisson cluster process, or to be a perturbation of a regular lattice.

Daley [1] discusses the problem of multidimensional renewal processes which he defines as processes constructed from one or more sets of independent and identically distributed variables and such that all parts of the plane are likely to contain points of the process. One such construction is the familiar one for a planar Poisson process with a constant parameter, which locates the events on a sequence of concentric circles, and uses two independent sequences of independent and

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identically distributed random variables, $\{A_i\}$ which are exponential and $\{\Theta_i\}$ which are uniform on $\{0,2\pi)$. For particular realisations $\{a_i\}$ and $\{\theta_i\}$ of these sequences, the polar coordinates of the points of the planar process are (r_i, θ_i) where $\pi r_i^2 = a_1 + \cdots + a_i$ $(i = 1,2,\ldots)$. In this paper some generalisations of this construction are considered which have one or more points on each of a sequence of concentric circles, and most of which satisfy Daley's definition.

In Section 2, the constructions considered have, like the Poisson process, just one event on each circle. Clearly, since the Poisson process is stationary, this type of process does not necessarily depend on the origin chosen for the centre of the set of concentric circles. In Section 3, constructions are discussed which have a random number of events on each concentric circle. In this case it ought to be possible to decide which point in the plane is the origin of the coordinate system by inspection of a realisation of the process, especially if the mean number of events on any circle is fairly large. The processes constructed are shown to be approximately Poisson far from the origin under certain sufficient conditions. In Section 4, the individual events of the processes considered in Section 3 are displaced independently along lines through the origin, in an attempt to remove some of the obvious regularity of these processes. Finally, in Section 5 the generalisation of these constructions to $\mathbb{R}^n$ is discussed.

2. Processes with one event on each circle

The constructions discussed in this Section resemble that for the Poisson process mentioned in the Introduction in that they use two sequences of random variables $\{A_i\}$ and $\{\Theta_i\}$ such that, for realisations $\{a_i\}$ and $\{\theta_i\}$ of these sequences, the events of the process have polar coordinates (r_i, θ_i) where $\pi r_i^2 = a_1 + \cdots + a_i$ $(i = 1,2,\ldots)$. The differences will come through the various assumptions which will be made about the sequences. For example, if $\{A_i\}$ and $\{\Theta_i\}$ are independent sequences of independent variables such that the A_i have an exponential distribution with parameter λ and the Θ_i have an arbitrary distribution $L(\cdot)$ over $[0,2\pi)$, then it is easy to show that the probability generating functional (p.g.fl) $G[\xi]$ for the constructed process is

$$\exp\left\{-\lambda\int_{\mathbb{R}^2} \mathrm{d}(\pi r^2)\mathrm{d}L(\theta)[1-\xi(r,\theta)]\right\}. \tag{2.1}$$

Of course, (2.1) is the p.g.fl for a Poisson process with a variable parameter. We note that, by definition, the argument ξ of the p.g.fl takes the value one outside some bounded region of the plane so that the integrand in (2.1) is non-zero only in a finite region of $\mathbb{R}^2$.

Suppose that we now wish to construct a process with a constant rate using the two sequences $\{A_i\}$ and $\{\Theta_i\}$ under the assumptions that these are independent sequences of independent variables and that the A_i have a common distribution function F satisfying $F(0)=0$. Then the only sequences possible are those which lead to a Poisson process. To see this denote the distribution function of Θ_i by L_i and the random counting measure of the process by N. Then

$$E(N(B)) = \lambda\,|B| \quad \text{for all bounded Borel sets } B \subset \mathbf{R}^2$$

if and only if

$$\sum_{n=1}^{\infty}\int_B \mathrm{d}F^{(n)}(\pi r^2)\mathrm{d}L_n(\theta) = \lambda\int_B \mathrm{d}(\pi r^2)\frac{1}{2\pi}\mathrm{d}\theta \tag{2.2}$$

where $F^{(n)}$ is the n-fold convolution of F. In particular, if B is a disc, centre the origin and with area x, then (2.2) becomes

$$\sum_{n=1}^{\infty} F^{(n)}(x) = \lambda x$$

which holds for all x only if F is the exponential distribution with parameter λ. Then, by substituting this form for F in (2.2), we obtain

$$\lambda\int_B \mathrm{d}(\pi r^2)\mathrm{e}^{-\lambda\pi r^2}\sum_{n=1}^{\infty}\frac{(\lambda\pi r^2)^{n-1}}{\Gamma(n)}\left[\mathrm{d}L_n(\theta)-\frac{1}{2\pi}\mathrm{d}\theta\right]=0. \tag{2.3}$$

The only solution of (2.3) is $L_n(\theta) = \theta/2\pi$. Thus the A_i must be exponential variables and the Θ_i must be uniformly distributed variables, so that the constructed process is Poisson.

More generally, it is clear from above that, if the nth process point has independent polar coordinates R_n, Θ_n such that πR_n^2 has a marginal distribution function $F^{(n)}$ which is the n-fold convolution of some distribution function F with $F(0-)=0$, then the rate of the process is a constant if and only if F is the exponential distribution and the angular coordinates Θ_n are marginally uniformly distributed. In particular we still obtain a process with a constant rate from the independent sequences $\{A_i\}$ and $\{\Theta_i\}$ if the A_i are independent exponential variables and the Θ_i are dependent but marginally uniformly distributed over $[0, 2\pi)$.

A simple form of dependency is for $\{\Theta_i\}$ to be a Markov sequence. To obtain second order properties of the constructed process, we need the conditional distribution of Θ_{i+j} given $\Theta_i = \theta_i$ for all i and j. One possibility for which this distribution is straightforward is to take Θ_1 to be uniformly distributed on $[0, 2\pi)$ and the density $l_j(\theta_{i+j}\mid\theta_i)$ of Θ_{i+j} conditional on $\Theta_i = \theta_i$ to be given by the cardioid density

$$l_j(\theta_{i+j}\mid\theta_i) = \frac{1}{2\pi}[1+2\rho^j\cos(\theta_{i+j}-\theta_i)] \quad (i, j = 1, 2, \ldots; |\rho| < 1/2)$$

from which it follows that Θ_i is marginally uniform on $[0, 2\pi)$ for all i. Consider this particular Markov sequence $\{\Theta_i\}$ together with a sequence $\{A_i\}$ of independent exponential variables with parameter λ. Then it is known that this process has a constant rate λ and that the number of events in any annulus with centre the origin is a Poisson variable. To investigate the second order properties of the process consider the measure

$$\Delta N(r,\theta)=N(\{(\rho,\phi): r\leqslant \rho< r+\Delta r, \theta\leqslant\phi<\theta+\Delta\theta\}).$$

Then if $r'>r$,

$$\begin{aligned}
&P\{\Delta N(r,\theta)\geqslant 1, \Delta N(r',\theta')\geqslant 1\}\\
&=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\lambda^2(\lambda\pi r^2)^{i-1}}{\Gamma(i)}\frac{[\lambda\pi(r'^2-r^2)]^{j-1}}{\Gamma(j)}e^{-\lambda\pi r'^2}\frac{1}{(2\pi)^2}(1+2\rho^j\cos(\theta'-\theta)]\times\\
&\qquad\times\Delta(\pi r^2)\Delta(\pi r'^2)\Delta\theta\Delta\theta'+o\{\Delta(\pi r^2)\Delta(\pi r'^2)\Delta\theta\Delta\theta'\} \qquad (2.5)\\
&=\lambda^2[1+2\rho\cos(\theta'-\theta)e^{-\lambda\pi(r'^2-r^2)(1-\rho)}]r\Delta r\Delta\theta r'\Delta r'\Delta\theta'+o\{r\Delta r\Delta\theta r'\Delta r'\Delta\theta'\}.
\end{aligned}$$

It follows from (2.5) that for any bounded Borel set $B\subset\mathbf{R}^2$,

$$\begin{aligned}
&V(N(B))=\\
&=\lambda|B|+2\lambda^2\int_{(r,\theta)\in B} r\,dr\,d\theta\int_{\{(r',\theta')\in B: r'>r\}} r'dr'd\theta' 2\rho\cos(\theta'-\theta)e^{-\lambda\pi(r'^2-r^2)(1-\rho)}. \qquad (2.6)
\end{aligned}$$

Suppose that B is a sector of an annulus, that is,

$$B=\{(r,\theta): R_1\leqslant r\leqslant R_2 \quad\text{and}\quad 0\leqslant\theta\leqslant\phi\}.$$

Then, since the process has a constant rate,

$$E(N(B))=\tfrac{1}{2}\lambda\phi(R_2^2-R_1^2),$$

while from (2.6) we see that

$$V(N(B))=\tfrac{1}{2}\lambda\phi(R_2^2-R_1^2)+\frac{2\rho(1-\cos\phi)}{\pi(1-\rho)^2}\{e^{-\lambda\pi(R_2^2-R_1^2)(1-\rho)}-1+\lambda\pi(R_2^2-R_1^2)(1-\rho)\}$$

showing that $N(B)$ is overdispersed or underdispersed with respect to a Poisson variable according to whether ρ is positive or negative. For an arbitrary set B, however, it would appear from (2.6) that the dispersion of $N(B)$ depends on the shape of B as well as on ρ.

Further generalisations of the Poisson process construction which might be discussed include a more complicated dependency within the $\{\Theta_i\}$, though it is hard to find a scheme which is compatible with the Θ_i being marginally uniformly distributed and thus the process having a constant rate. Also, the sequence $\{A_i\}$ could be a sequence of non-identically distributed or dependent variables. In this case the rate of the process is unlikely to be constant. Another possibility is to let the $\{A_i\}$ and $\{\Theta_i\}$ sequences be dependent. For example, a simple case would be to let the A_i be independent exponential variables with parameter λ and then define $\Theta_i = a_i$ modulo 2π. Even this does not give a tractable process.

3. Processes with random numbers of events on each circle

The processes considered in this Section have a random number of events on each of a sequence of concentric circles. We shall discuss only the following very simple situation. Let $\{A_i\}$ be a sequence of independent exponentially distributed variables with parameter λ. Let $\{M_i\}$ be a sequence of independent non-negative integer-valued random variables all with probability generating function (p.g.f.) $Q(z) = \sum_{n=0}^{\infty} q_n z^n$, for which $E(M_i) = \sum_{n=0}^{\infty} nq_n < \infty$, and denote by $\{m_i\}$ a realisation of this sequence. Given $\{m_i\}$, let $\{\Theta_{ij} : j = 1,\dots, m_i\,;\, i = 1,2,\dots\}$ be a sequence of independent variables all with distribution function L over $[0, 2\pi)$. Assume that $\{A_i\}$, $\{\Theta_{ij}\}$ and $\{M_i\}$ are independent sequences and that realisations of $\{A_i\}$ and $\{\Theta_{ij}\}$ are denoted by $\{a_i\}$ and $\{\theta_{ij}\}$ respectively. Then the point process in $\mathbf{R}^2$ which we consider has the realisation $\{(r_i, \theta_{ij}) : j = 1,\dots, m_i\,;\; i = 1,2,\dots\}$ where $\pi r_n^2 = a_1 + \cdots + a_n$, $(n = 1,2,\dots)$. We shall denote the counting measure of this process by N and the p.g.fl by $G[\xi]$.

The variables A_i are chosen to be exponential since this was found to be the most tractable case in the situation considered in the last Section in which $M_i = 1$ with probability one. It is clear that without loss of generality we may assume that $q_0 = 0$.

It is straightforward to obtain the p.g.fl for the process:

$$G[\xi] = \exp\left\{ -\lambda \int_{r=0}^{\infty} \mathrm{d}(\pi r^2)[1 - Q(\bar{\xi}(r))] \right\} \tag{3.1}$$

where $\bar{\xi}(r) = \int_0^{2\pi} \mathrm{d}L(\theta)\xi(r, \theta)$. If, for some bounded Borel set $A \subset \mathbf{R}^2$, the substitution

$$\xi(r, \theta) = \begin{cases} z, & \text{if } (r, \theta) \in A, \\ 1, & \text{otherwise,} \end{cases} \tag{3.2}$$

is made in (3.1) then $G[\xi]$ becomes the p.g.f. of $N(A)$. Now if ξ is given by (3.2) it follows that $\bar{\xi}(r)$ may be written as $1-(1-z)\alpha(r)$ where $0 \le \alpha(r) \le 1$, and then

$$1 - Q(\bar{\xi}(r)) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} q_n \binom{n}{k} (-1)^{k-1}(1-z)^k \alpha^k(r).$$

Thus we see that the first two moments of $N(A)$ are given by

$$E(N(A)) = \lambda E(M) \int_{r=0}^{\infty} \mathrm{d}(\pi r^2)\alpha(r), \tag{3.3}$$

$$V(N(A)) = \lambda E(M) \int_{r=0}^{\infty} \mathrm{d}(\pi r^2)\alpha(r) + \lambda E[M(M-1)] \int_{r=0}^{\infty} \mathrm{d}(\pi r^2)\alpha^2(r), \tag{3.4}$$

where M denotes a variable with p.g.f. Q.

From (3.3) and (3.4) it follows that $N(A)$ is overdispersed with respect to a Poisson process for all bounded Borel sets A as long as $E[M(M-1)] > 0$, that is, there is a positive probability of there being more than one event on any circle. Since we took $q_0 = 0$, $E[M(M-1)] = 0$ only if $M = 1$ with probability one and in this case it has already been shown in Section 2 that the process is Poisson.

As a simple example suppose that A is a sector of an annulus, say

$$A = \{(r, \theta) : R_1 \leq r \leq R_2 \quad \text{and } 0 \leq \theta \leq \phi\}.$$

Then

$$\alpha(r) = \begin{cases} L(\phi) & \text{if } R_1 \leq r \leq R_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $L(\phi)$ denotes $L[0, \phi]$. Then we obtain

$$\begin{aligned} E(z^{N(A)}) &= \exp\{-\lambda\pi(R_2^2 - R_1^2)[1 - Q(1-(1-z)L(\phi))]\}, \\ E(N(A)) &= \lambda\pi(R_2^2 - R_1^2)L(\phi)E(M), \\ V(N(A)) &= \lambda\pi(R_2^2 - R_1^2)L(\phi)\{E(M) + E[M(M-1)]L(\phi)\}. \end{aligned} \tag{3.5}$$

As another simple case, suppose that L is the uniform distribution on $[0, 2\pi)$. Then the function $\alpha(r)$ is

$$\alpha(r) = \frac{1}{2\pi r} \times l \tag{3.6}$$

(where l is the length of intersection of a circle, centre the origin and radius r, with the set A) and

$$\int_{r=0}^{\infty} \mathrm{d}(\pi r^2)\alpha(r) = |A|.$$

In this situation, clearly all the properties of the process will be invariant under rotations of the sets involved about the origin. Because of the form of $\alpha(r)$, given in (3.6), it is straightforward to obtain the variance $V(N(D))$ where D is any disc with radius R centered on some arbitrary point of the plane. Since the algebra is, in general, rather tedious we consider the situation in which the origin is on the perimeter of the disc. Then

$$\alpha(r) = \begin{cases} \dfrac{1}{\pi}\cos^{-1}(r/2R), & r \leq 2R \\ 0 & r > 2R. \end{cases}$$

Therefore

$$\int_{r=0}^{\infty} \mathrm{d}(\pi r^2)\alpha^2(r) = \frac{R^2}{2\pi}(\pi^2 - 4)$$

and thus

$$E(N(D)) = \lambda E(M)\pi R^2,$$

$$V(N(D)) = \lambda E(M)\pi R^2 + \lambda E[M(M-1)]\frac{R^2}{2\pi}(\pi^2 - 4). \tag{3.7}$$

If in (3.5) we put $R_1 = 0$, $\phi = 2\pi$ and $R_2 = R$ and then compare that expression for $V(N(A))$ with the form for $V(N(D))$ in (3.7), we see that the variance for the number of events in a disc D, radius R, which has its centre a distance R from the origin, is less than or equal to the variance for the disc A with radius R and centre the origin. There is equality only if $M = 1$ with probability one: the Poisson process case. In fact, it can be shown that the variance of the number of events in a disc, radius R and centred a distance d from the origin, is a monotonic decreasing function of d.

Finally in this Section the properties of the process far from the origin are considered. It seems intuitively reasonable, at least if L is the uniform distribution and perhaps for other well-behaved distributions, that the process will be asymptotically Poisson. This result can almost certainly be proved using one of the limit theorems which exist in the literature. However, this asymptotic property can also be demonstrated using direct methods.

To show that the process is asymptotically Poisson we have to show that all the finite-dimensional distributions for the process converge to the corresponding distributions for a Poisson process as the sets involved in the distributions are translated to infinity. Clearly the parameter of the process will be $\lambda E(M)$ if L is the uniform distribution. Since the Borel sets in the plane can be generated by the rectangles, we need only consider the joint distributions of the numbers of process points in disjoint rectangles, where the sides of the rectangles are taken to be parallel and perpendicular to the direction of translation.

To start with we consider just two such rectangles A and B and assume that L is the uniform distribution on $[0, 2\pi)$. Let $t > 0$. Then the joint p.g.f. of $N(A+t)$ and $N(B+t)$, where $A + t = \{x + t\,;\, x \in A\}$, is derived by the substitution

$$\xi(r,\theta) = \begin{cases} z_a & \text{if } (r,\theta) \in A + t, \\ z_b & \text{if } (r,\theta) \in B + t, \\ 1 & \text{otherwise} \end{cases}$$

in (3.1) The function $\bar{\xi}(r)$ can then be obtained, and if $1 - Q(\bar{\xi}(r))$ is written as a sum, the integral with respect to r can be performed term by term. If the limit as $t \to \infty$ is taken, again a term by term procedure is valid, this leads to the result

$$E(z_a^{N(A+t)} \cdot z_b^{N(B+t)}) \to \exp\{-\lambda E(M)[(1 - z_a)|A| + (1 - z_b)|B|]\},$$

which shows that $N(A+t)$ and $N(B+t)$ converge in distribution, as $t \to \infty$, to independent Poisson variables.

In an entirely similar way, if $A_1, \ldots, A_k$ are disjoint rectangles for any positive integer k, then $N(A_1 + t), \ldots N(A_k + t)$ can be shown to converge in distribution, as $t \to \infty$, to independent Poisson variables. Thus the process is asymptotically Poisson, with parameter $\lambda E(M)$.

It only remains to consider what happens to this result when the angle sequence has a more general distribution L on $[0, 2\pi)$. Suppose that L is absolutely continuous with a density function l such that for $\varepsilon \geq 0$, $l(2\pi - \varepsilon) \to l(0)$ as $\varepsilon \to 0$, and such that, defining $l(2\pi)$ to be $l(0)$, l is a continuous function on $[0, 2\pi]$. Then it is intuitively clear, and it can be proved in the same way as when L is the uniform distribution, that under these circumstances the process is asymptotically Poisson with a variable parameter. That is, if the limit is taken as the sets are translated to infinity along the direction θ, then the parameter of the Poisson process is $\lambda E(M) 2\pi l(\theta)$. If the distribution L is not as well-behaved as this then the problem becomes more complicated. For example, suppose that L has an atomic component as well as an absolutely continuous one. Then, in order to get straightforward results the direction in which the sets are translated must be neither an atom of L nor a limit point of atoms of L. In this case, if the density corresponding to the absolutely continuous component of L is continuous and bounded in some neighbourhood of the chosen direction, then the above result still holds.

4. Displacement of the concentric circle process

Some of the regularity of the process discussed in Section 3 can be removed by displacing independently the points off the concentric circles. Let $D(r', \theta'/r, \theta)$ be the probability that a point with polar coordinates (r, θ) is displaced into the set $\{(\rho, \phi): 0 \leq \rho \leq r', 0 \leq \phi \leq \theta'\}$. Then, if the points of the process which has p.g.fl given by (3.1) are displaced independently according to the distribution D, the displaced process will have the p.g.fl $G_d[\xi]$ given by

$$G_d[\xi] = \exp\left\{-\lambda \int_0^\infty d(\pi r^2)[1 - Q(\eta(r))]\right\} \tag{4.1}$$

where

$$\eta(r) = \int_0^{2\pi} dL(\theta) \int_{\mathbf{R}^2} dD(r', \theta'/r, \theta)\xi(r', \theta').$$

The process of Section 3 was shown to be asymptotically Poisson if L is suitably behaved. It thus follows that its displacement is asymptotically Poisson as long as

$$\int_0^\infty d(\pi r^2) \int_0^{2\pi} dL(\theta) dD(r', \theta'/r, \theta)$$

is a Borel measure.

Specifically, suppose that the random point (R, Θ) is displaced to (R', Θ) such that either (a) $R' = X_d + R$ or (b) $R' = X_d R$, where X_d is a non-negative variable independent of R with distribution function F_d. To get the p.g.f. of the number of events of the displaced process which lie in A, $N_d(A)$ say, substitute

$$\xi(r,\theta)=\begin{cases} z & (r,\theta)\in A \\ 1 & \text{otherwise} \end{cases}$$

in (4.1) and (4.2). If, as in Section 3, $\alpha(r)$ is then defined by $\bar{\xi}(r)=1-(1-z)\alpha(r)$, it is simply shown that either

$$\text{(a)} \qquad \eta(r)=1-(1-z)\int_0^\infty dF_d(x)\alpha(x+r),$$

or

$$\text{(b)} \qquad \eta(r)=1-(1-z)\int_0^\infty dF_d(x)\alpha(xr).$$

Thus the first two moments of $N_d(A)$ are given by (3.3) and (3.4), which are the corresponding moments for the non-displaced process, if we replace $\alpha(r)$ by either $\int_0^\infty dF_d(x)\alpha(x+r)$ in case (a), or $\int_0^\infty dF_d(x)\alpha(xr)$ in case (b), throughout these equations.

Consequently it follows that $N_d(A)$ is overdispersed with respect to the Poisson process as long as $\mathbf{P}\{M=1\}<1$. If $\mathbf{P}\{M=1\}=1$ then the non-displaced process is Poisson and therefore $N_d(A)$ will have a unit index of dispersion for all bounded Borel sets $A\subset\mathbf{R}^2$. However, if $\mathbf{P}\{M=1\}=1$, then (4.1) becomes either

$$\text{(a)} \quad G_d[\xi]=\exp\left\{-\lambda\int_0^\infty d(\pi r^2)\int_0^{2\pi} dL(\theta)\left[1-\int_0^\infty dF_d(x)\xi(x+r,\theta)\right]\right\}$$

or

$$\text{(b)} \quad G_d[\xi]=\exp\left\{-\lambda\int_0^\infty d(\pi r^2)\int_0^{2\pi} dL(\theta)\left[1-\int_0^\infty dF_d(x)\xi(xr,\theta)\right]\right\}$$

from which it is clear that the displaced process is not Poisson except in the degenerate case when the distribution of R' conditional on $R=r$ has a unit atom at r, that is, in case (a) F_d has all its weight at the origin, while in case (b) F_d has all its weight at the unit value. (Note that in neither case are the displacements of the events independent and identically distributed.)

Finally we note that since the choice of a particular distribution function F_d, for example the exponential distribution, does not seem to simplify appreciably the results for the displaced process, no properties of any special cases are given here.

5. Generalisations to $\mathbf{R}^n$

The constructions discussed in this paper generalise to $\mathbf{R}^n$ in a straightforward way. In $\mathbf{R}^n$ we have the following situation. The polar coordinates of a point are $(r,\boldsymbol{\theta})$ where $r\in[0,\infty)$ and $\boldsymbol{\theta}$ is an $(n-1)$-dimensional vector such that $\theta_i\in[0,\pi)$ $(i=1,\ldots,n-2)$ and $\theta_{n-1}\in[0,2\pi)$. The volume element dV is given by

$$dV = r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, dr\, d\theta_1 \cdots d\theta_{n-1}$$

while the element of solid angle, $d\Omega$, is

$$d\Omega = dV/r^{n-1}dr.$$

The volume of a ball of radius r is

$$V = c_n r^n \quad \text{where } c_n = \pi^{n/2}/\Gamma(\tfrac{1}{2}(n+2))$$

and the surface area is

$$S = 2\pi^{n/2} r^{n-1}/\Gamma(\tfrac{1}{2}n).$$

Thus we may generalise the construction of a Poisson process to $\mathbb{R}^n$ as follows. Let $\{A_i\}$ be, as before, a sequence of independent exponential variables with parameter λ and let a sequence $\{R_i\}$ be defined by

$$c_n R_i^n = A_1 + \cdots + A_i.$$

Let $\{\Theta_i = (\Theta_{1i}, \ldots, \Theta_{n-1\,i})\}$ be an independent sequence of independent vector-valued variables with common distribution function $L(\theta)$, such that Θ_{ji} takes values in $[0, \pi)$ for $j = 1, \ldots, n-2$ and $\Theta_{n-1\,i}$ takes values in $[0, 2\pi)$. Consider the point process which consists of the sequence of points $\{(R_i, \Theta_i)\ i = 1, 2, \ldots\}$. It is straightforward to write down the p.g.fl $G[\xi]$ for this process, and it is given by

$$G[\xi] = \exp\left\{-\lambda \int_{\mathbb{R}^n} d(c_n r^n)dL(\theta)\,[1 - \xi(r, \theta)]\right\} \tag{5.1}$$

which reduces to (2.1) if $n = 2$.

This is the p.g.fl of a Poisson process with a variable parameter. We note that if the Poisson process is to have a constant parameter (λ) then $L(\theta)$ must be such that

$$d(c_n r^n)dL(\theta) = dV,$$

that is,

$$dL(\theta) = \frac{\Gamma(\tfrac{1}{2}n)}{2\pi^{n/2}} d\Omega.$$

This is to be expected intuitively. It means that having fixed the radial coordinate r of the point, the angular coordinate θ is chosen in such a way that the point is uniformly distributed over the surface of a ball with radius r. In this case the components of Θ_i are independent variables, and $\Theta_{n-1\,i}$ is uniformly distributed over $[0, 2\pi)$ while $\Theta_{n-j\,i}$ has density

$$\frac{\Gamma(\tfrac{1}{2}(j+1))}{\pi^{1/2}\Gamma(\tfrac{1}{2}j)} \sin^{j-1}\theta, \quad j = 2, \ldots, n-1.$$

As for processes in $\mathbb{R}^2$, if the ith process point has polar coordinates (R_i, Θ_i) such that R_i and Θ_i are independent and where the marginal distribution of $c_n R_i^n$ is the i-fold convolution $F^{(i)}$ of some distribution function F with $F(0-) = 0$, then the rate of the process is a constant only if F is the exponential distribution and the Θ_i have a common marginal distribution given by

$$\frac{\Gamma(\frac{1}{2}n)}{2\pi^{\frac{1}{2}n}}\,\mathrm{d}\Omega.$$

In Sections 3–4 where there are a random number of events with the same radial coordinate the results apply if the obvious modifications are made. For example, $\bar{\xi}(r)$ has to be defined as

$$\bar{\xi}(r)=\int \mathrm{d}L(\theta)\xi(r,\theta)$$

where $L(\theta)$ is the distribution of the angular coordinate Θ and the integral is over the region $[0,\pi)^{n-2}\times[0,2\pi)$, and then

$$\eta(r)=\int \mathrm{d}L(\theta)\int_{(r',\theta')\in R^n} \mathrm{d}D(r',\theta'\mid r,\theta)\xi(r',\theta').$$

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